Adaptive Robust Transmission Network Expansion Planning using Structural Reliability and Decomposition Techniques

R. Mínguez[a,*], R. García-Bertrand[b], J. M. Arroyo[b]

[a] *Independent consultant, C/Honduras 1, 13160, Torralba de Calatrava, Ciudad Real, Spain*

[b] *Department of Electrical Engineering, University of Castilla-La Mancha, Ciudad Real, Spain*

Abstract

Structural reliability and decomposition techniques have recently proved to be appropriate tools for solving robust uncertain mixed-integer linear programs using ellipsoidal uncertainty sets. In fact, its computational performance makes this type of problem to be an alternative method in terms of tractability with respect to robust problems based on cardinality constrained uncertainty sets. This paper extends the use of these techniques for solving an adaptive robust optimization (ARO) problem, i.e. the adaptive robust solution of the transmission network expansion planning for energy systems. The formulation of this type of problem materializes on a three-level mixed-integer optimization formulation, which based on structural reliability methods, can be solved using an ad-hoc decomposition technique. The method allows the use of the correlation structure of the uncertain variables involved by means of their variance-covariance matrix, and besides, it provides a new interpretation of the robust problem based on quantile optimization. We also compare results with respect to robust optimization methods that consider cardinality constrained uncertainty sets. Numerical results on an illustrative example, the IEEE-24 and IEEE 118-bus test systems demonstrate that the algorithm is comparable in terms of computational performance with respect to existing robust methods with the additional advantage that the correlation structure of the uncertain variables involved can be incorporated

*Corresponding author: rominsol@gmail.com, tlfn.: 00 34 926810046

straightforwardly.

---

1. Introduction

Robust optimization techniques were developed to deal with the problem of designing solutions that are immune to data uncertainty by solving equivalent deterministic problems, i.e. robust counterparts (RC). The first published work about these techniques is proposed by Soyters [1], which allowed uncertain parameters to take their worst possible values within a given interval. However, although using this method the RC of any linear mathematical programming problem remains linear, Soyters's solutions are excessively conservative, and [2, 3, 4] proposes the use of alternative uncertainty sets, the ellipsoidal uncertainty sets based on the Mahalanobis distance. This alternative method allows controlling the grade of conservatism, and includes straightforwardly the correlation structure of the uncertain parameters involved by means of the variance-covariance matrix. In contrast, the RC derived from any linear mathematical programming problem using ellipsoidal uncertainty sets results in a conic quadratic problem. This feature, which is not problematic for those cases where interior point algorithms are applicable because they have been proved to be solvable in polynomial time, is not particularly attractive for solving robust linear discrete optimization models, as pointed out by [5]. It is precisely [5] the one that proposes the use of an alternative uncertainty set that allows the RC problem to remain as linear, even if binary variables are involved. Instead of allowing all uncertain parameters to take their worst possible values within the given interval such as [1], it only allows a pre-established maximum number of parameters Γ to do so. This alternative formulation also provides a robust solution in terms of probability of infeasibility. Recently, [6] proposes an alternative and decomposable solution technique inspired on structural reliability First-Order-Second-Moment (FOSM) methods, which allows reaching the optimal solution of the RC problem using ellipsoidal uncertainty sets by solving two kind of problems within an iterative scheme: one master problem of the same size as the original RC problem, and one subproblem of considerable lower size for each uncertain constraint. The method proposed by [6] has the following advantages:

1. It improves tractability with respect to interior point algorithms, specially for large-scale problems and those including binary decision vari-

ables.

2. In addition and due to its relationship with reliability-based structural techniques, it allows to calculate exact bounds for the probability of constraint violation in case the probability distributions of uncertain parameters are completely defined by using first and second moments (mean and variance-covariance).
3. In terms of tractability, it constitutes an alternative formulation, specially for problems including binary variables, to that proposed by [5] using cardinality constrained uncertainty sets.

Note that the formulation proposed by [6] was tested and proved to be successful on mixed-integer linear mathematical programming problems with hard constraints that must be satisfied for any possible realization of the uncertain data. This type of problems is relevant for optimal design, however, problems with recourse [7, 8, 9] where the decision-maker must take some decisions before discovering the actual value of the uncertain parameters and having the opportunity to take further action once the uncertain parameters have been revealed, are more common on engineering problems and can not take advantage of the features of the method proposed by [6] in its present form. This framework is also known as Adaptive Robust Optimization (ARO).

A clear example of the practical importance of robust linear optimization with recourse (ARO) is the Transmission Network Expansion Planning (TNEP) problem. It aims to resolve the issue of how to expand or reinforce an existing electricity transmission network so that adequately serves system loads over a given horizon. The main difficulty of this problem is to take decisions under the great amount of uncertainty associated with i) demands, ii) renewable generation, such as wind and solar power plants, and iii) equipment failure. There are several examples of the successful application of robust optimization in TNEP problems based on cardinality constrained uncertainty sets proposed by [5], such as, [10, 11, 12, 13, 14, 15]. These works use three-level formulations where first level minimizes the cost of expansion ([13] also minimizes the maximum regret), the decision variables for this level are those associated with construction or expansion of lines, ii) second level selects the realization of the uncertain parameters that maximizes the system's operating costs within the uncertainty set, the variables related to this level are the uncertain parameters, i.e., generation capacity and demand, and iii) in the third level the system operator selects the optimal decision

variables to minimize operating costs for given values of first and second level variables.

This paper proposes a double iterative solution approach to the three-level mixed-integer optimization problem associated with TNEP. The idea is to replace cardinality constrained by ellipsoidal uncertainty sets, which allows incorporating the correlation structure of the uncertain parameters straightforwardly by means of the variance-covariance matrix. Inner level problems (subproblems) are solved throughout an iterative method inspired from structural reliability and decomposition techniques [6]. Note that, due to the way the uncertainty sets are constructed, the subproblem does not contain binary variables. Meanwhile, first level problem (master problem) uses the constraint-and-column generation method (proposed by [16]) initially used in this type of problems by [13] and [14], and also utilized by [15].

The proposed method has the following advantages which might make it attractive for practical use:

1. Computational performance is comparable with the method proposed by [15] based on cardinality constrained uncertainty sets. Note that this method proved to be the most efficient method to date for solving this type of problem.
2. The subproblem can be solved using two nested linear optimization problems: i) the first one takes the optimal operating decisions for given realizations of the uncertain parameters involved, and ii) the second one updates the worst possible realization of the uncertain parameters in terms of operating costs. This second problem, under certain circumstances, have analytical solutions and otherwise can be solved efficiently by using interior point algorithms.
3. The approach proposed in this paper requires the definition of the protection level, which establishes approximately the quantile of the uncertain objective function to be optimized. Besides, the proposed method would not require the definition of bounds for the uncertain parameters, which might be of interest for certain cases.

The remainder of the paper is structured as follows. Section 2 describes the adaptive robust formulation of the TNEP problem in compact matrix form, and discusses the solution approach proposed by [15] inspired in the works by [12, 13, 14]. In Section 3 the definition of the uncertainty set and the description of the solution approach are presented. Numerical results

for different networks are given in Section 4 and compared with those obtained using the method proposed by [15]. Finally, the paper is concluded in Section 5.

2. Transmission network expansion planning problem: ARO compact formulation

Note that for clarity in the exposition, we adopt the same nomenclature used by [15]. Thus, the robust TNEP problem can be written in compact matrix form as the following three-level mathematical programming problem:

$$\underset{\boldsymbol{x}}{\text{Minimize}} \quad \left(\boldsymbol{c}^T\boldsymbol{x} + \underset{\boldsymbol{d} \in \boldsymbol{D}}{\text{Maximum}} \quad \underset{\boldsymbol{y} \in \Omega(\boldsymbol{x},\boldsymbol{d})}{\text{Minimum}} \quad \boldsymbol{b}^T \ \boldsymbol{y} \right) \tag{1}$$

subject to

$$\begin{aligned} \boldsymbol{c}^T\boldsymbol{x} \ &\leq \ \Pi & (2)\\ \boldsymbol{x} \ &\in \ \{0,1\}, & (3) \end{aligned}$$

where $\boldsymbol{x}$ is the vector of first stage binary variables representing the *investment* vs *no investment* in reinforcing or building new lines, $\boldsymbol{c}$ is the investment cost vector, $\boldsymbol{d}$ is the vector of second stage continuous variables representing the random or uncertain parameters, i.e. generation capacities and demands, $\boldsymbol{D}$ is the uncertainty set, $\boldsymbol{b}$ is the vector including operating costs, and $\boldsymbol{y}$ is the vector of third stage continuous variables referring to the operating variables. These operating variables include powers consumed, power flows, power produced by generating units, load shed by demands, and voltage angles at buses (see reference [15] for a detail description of the formulation). Π represents the maximum budget for investment in transmission expansion. Finally, $\Omega(\boldsymbol{x},\boldsymbol{d})$ defines the feasibility region for the operating variables $\boldsymbol{y}$, as a function of investment decisions $\boldsymbol{x}$ and given realizations of the uncertain parameters $\boldsymbol{d}$, as follows:

$$\Omega(\boldsymbol{x},\boldsymbol{d}) = \left\{ \begin{array}{rcl} \boldsymbol{Ax} + \boldsymbol{By} & = & \boldsymbol{E} : \boldsymbol{\lambda} \\ \boldsymbol{Fx} + \boldsymbol{Gy} & \leq & \boldsymbol{K} : \boldsymbol{\mu} \\ \boldsymbol{I}_{\text{eq}}\boldsymbol{y} & = & \boldsymbol{d} : \boldsymbol{\alpha} \\ \boldsymbol{I}_{\text{ineq}}\boldsymbol{y} & \leq & \boldsymbol{d} : \boldsymbol{\varphi}, \end{array} \right. \tag{4}$$

where $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{E}$, $\boldsymbol{F}$, $\boldsymbol{G}$ and $\boldsymbol{K}$ are matrices of constant parameters dependent on the network configuration and element characteristics, $\boldsymbol{I}_{\text{eq}}$ selects

the components of $\boldsymbol{y}$ that are equal to the uncertain parameters (demands), and $\boldsymbol{I}_{\text{ineq}}$ selects the components of $\boldsymbol{y}$ that are limited by the uncertain parameters (i.e. maximum power generation and maximum load shedding). The first set of equality constraints correspond to constraints enforcing the power balance at every bus, the power flow through each line, and fixing the voltage angle of the reference bus. The first set of inequality constraints are associated with line flow limits, and limits on the voltage angles at every bus. Note that $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\varphi}$ are the dual variable vectors associated with each set of constraints, respectively.

For a detailed physical interpretation of the mathematical formulation (1)-(3) we recommend the paper by [14].

2.1. Bi-level approach proposed by [15]

[15], analogously to [12], proposed dealing with problem (1)-(3) by decomposing and iteratively solving a subproblem and a master problem. The master variables correspond to $\boldsymbol{x}$, i.e. the vector of first stage binary variables. Thus, for given values of these master variables, the subproblem corresponds to:

$$\underset{\boldsymbol{d} \in \boldsymbol{D}}{\text{Maximum}} \quad \underset{\boldsymbol{y} \in \Omega(\boldsymbol{x}, \boldsymbol{d})}{\text{Minimum}} \quad \boldsymbol{b}^T \boldsymbol{y} \ . \tag{5}$$

Considering the dual problem associated with the third-level, (5) results in the following single level maximization problem:

$$\underset{\boldsymbol{d}, \boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\alpha}, \boldsymbol{\varphi}}{\text{Maximize}} \quad f^{\text{dual}} = (\boldsymbol{E} - \boldsymbol{A}\boldsymbol{x})^T \boldsymbol{\lambda} - (\boldsymbol{K} - \boldsymbol{F}\boldsymbol{x})^T \boldsymbol{\mu} + \boldsymbol{d}^T (\boldsymbol{\alpha} - \boldsymbol{\varphi}) \tag{6}$$

subject to

$$\begin{aligned} \boldsymbol{B}^T \boldsymbol{\lambda} - \boldsymbol{G}^T \boldsymbol{\mu} + \boldsymbol{I}_{\text{eq}}^T \boldsymbol{\alpha} - \boldsymbol{I}_{\text{ineq}}^T \boldsymbol{\varphi} &= \boldsymbol{b} && (7) \\ \boldsymbol{\mu} &\geq \boldsymbol{0} && (8) \\ \boldsymbol{\varphi} &\geq \boldsymbol{0}, && (9) \end{aligned}$$

$$\boldsymbol{d} \in \boldsymbol{D}. \tag{10}$$

Regarding the uncertainty set $\boldsymbol{D}$, [15] proposed the use of the following

set:

$$
\begin{aligned}
d_i^G &= \bar{d}_i^G - \hat{d}_i^G z_i^G;\ \forall i \in \Omega^{\mathrm{G}} && (11)\\
d_i^D &= \bar{d}_i^D + \hat{d}_i^D z_i^D;\ \forall i \in \Omega^{\mathrm{D}} && (12)\\
\sum_{i \in (\Omega^{\mathrm{G}} \cap\, \Omega^{\mathrm{r}})} z_i^G &\leq \Gamma_r^G;\ \forall r && (13)\\
\sum_{i \in (\Omega^{\mathrm{D}} \cap\, \Omega^{\mathrm{r}})} z_i^D &\leq \Gamma_r^D;\ \forall r && (14)\\
z_i^G &\in \{0,1\}; \forall i \in \Omega^{\mathrm{G}} && (15)\\
z_i^D &\in \{0,1\}; \forall i \in \Omega^{\mathrm{D}}, && (16)
\end{aligned}
$$

where d_i^G is the uncertain generation limit related to the ith variable within vector $\boldsymbol{d}$, $\bar{d}_i^G$ is the corresponding nominal value, $\hat{d}_i^G$ is the maximum positive distance from the nominal value that can take the random parameter, and Ω^{G} is the set of indices of the generating units. Analogously, d_i^D, $\bar{d}_i^D$, $\hat{d}_i^D$ and Ω^{D} correspond to the same values but for demands. The only limitation introduced by this simplified set is that uncertainty budgets Γ_r^G and Γ_r^D must be integer values, which in our opinion does not dwarf the benefits of robust optimization from the practical perspective. Alternatively, instead of uncertainty budgets associated with generation, demand and regions Γ_r^G and Γ_r^D, a unique uncertainty for each region Γ_r, or for the system Γ could be used instead.

Note that this set takes advantage of the simplification proposed by [17] because the uncertain parameters might only take either the nominal, or upper or lower limits of their uncertainty range, and the fact noticed by [14] that the worst realization of "nature" try to make generation capacities to be as lower as possible and demand loads as higher as possible. This set contains a reduced number of binary variables with respect to [12], which reduce the number of binary variables by half, increasing the percentage of reduction with respect to [13] and [14].

Subproblem (6)-(10) is a bilinear mathematical programming problem, which can be linearized and transformed into a mixed-integer linear mathematical programming problem at the expense of introducing binary variables associated with the uncertainty set (for more details, see the transformation of $\boldsymbol{d}^T(\boldsymbol{\alpha} - \boldsymbol{\varphi})$ introduced by [15]).

The optimal solution of subproblem (6)-(10) provides the uncertain parameter values $\boldsymbol{d}_{(i)}$ to construct primal cuts for the master problem, which

corresponds to the following optimization problem at iteration k:

$$\underset{\boldsymbol{x}, \boldsymbol{y}_{(i)};\ \forall i=1,\ldots,k-1}{\text{Minimize}} \quad \boldsymbol{c}^T \boldsymbol{x} + \gamma \tag{17}$$

subject to

$$\begin{aligned}
\gamma &\geq \boldsymbol{b}^T \boldsymbol{y}_{(i)};\ \forall i = 1, \ldots, k-1 && (18)\\
\gamma &\geq 0 && (19)\\
\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{y}_{(i)} &= \boldsymbol{E};\ \forall i = 1, \ldots, k-1 && (20)\\
\boldsymbol{F}\boldsymbol{x} + \boldsymbol{G}\boldsymbol{y}_{(i)} &\leq \boldsymbol{K};\ \forall i = 1, \ldots, k-1 && (21)\\
\boldsymbol{I}_{\text{eq}}\boldsymbol{y}_{(i)} &= \boldsymbol{d}_{(i)};\ \forall i = 1, \ldots, k-1 && (22)\\
\boldsymbol{I}_{\text{ineq}}\boldsymbol{y}_{(i)} &\leq \boldsymbol{d}_{(i)};\ \forall i = 1, \ldots, k-1, && (23)\\
\boldsymbol{c}^T \boldsymbol{x} &\leq \Pi && (24)\\
\boldsymbol{x} &\in \{0, 1\}. && (25)
\end{aligned}$$

Note that the master problem, besides variable γ, includes one operating variable vector $\boldsymbol{y}_{(i)}$ for each realization of the uncertain parameters obtained from subproblem (6)-(10).

3. Proposed solution technique

This section justifies and explain in detail the proposed solution technique. However, before focusing on the mathematical description of the solution method, we justify the type of uncertainty set used in this paper.

3.1. Ellipsoidal uncertainty set

Analogously to [3] and [6], we propose the following uncertainty set. Let us assume that uncertain parameter vector $\boldsymbol{d} = (\boldsymbol{d}^{\text{G}},\ \boldsymbol{d}^{\text{D}})^T$ has nominal or expected values $\bar{\boldsymbol{d}}$ and variance-covariance matrix $\boldsymbol{\Sigma}$, respectively. According to [3], the ellipsoidal uncertainty set can be written using the Mahalanobis distance as follows:

$$\boldsymbol{D}(\beta) = \left\{ \left(\boldsymbol{d} - \bar{\boldsymbol{d}}\right)^T (\boldsymbol{\Sigma})^{-1} \left(\boldsymbol{d} - \bar{\boldsymbol{d}}\right) \leq \beta^2 \right\}, \tag{26}$$

which constitutes the uncertainty set in (10). In contrast with the method proposed by [15], one unique parameter β controls the size and protection level of the ellipsoidal set.

Note that only first and second order moments of the random parameters without lower and upper bounds are initially considered in this paper for the ellipsoidal uncertainty set. There are two reasons, firstly, the simplification of the process as it will be shown in next subsections, and secondly, because it is straightforward to include also the interval limitations $\left(\bar{\boldsymbol{d}} - \hat{\boldsymbol{d}}, \bar{\boldsymbol{d}} + \hat{\boldsymbol{d}}\right)$ of the random parameters within the proposed solution method.

In order to simplify the problem, the uncertainty set (26) is transformed using an affine mapping into a ball of radius β, resulting in the following set of alternative constraints:

$$\boldsymbol{d} = \bar{\boldsymbol{d}} + \boldsymbol{L}\boldsymbol{z}; \ \|\boldsymbol{z}\| \leq \beta, \tag{27}$$

where $\boldsymbol{L}$ is the mapping matrix which can be obtained from Cholesky decomposition of variance-covariance matrix $\boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{L}^T$, $\boldsymbol{z}$ represents a perturbation vector and $\|\cdot\|$ stands for Euclidean norm. Note that using also the fact noticed by [14] that the worst realization of "nature" try to make generation capacities to be as lower as possible and demand loads as higher as possible, we could reduce the uncertainty set by half including the following additional constraints:

$$\boldsymbol{d}^{\mathrm{G}} \leq \bar{\boldsymbol{d}}^{\mathrm{G}} \tag{28}$$

$$\boldsymbol{d}^{\mathrm{D}} \geq \bar{\boldsymbol{d}}^{\mathrm{D}}, \tag{29}$$

which force maximum power productions and demand loads to take, respectively, values below and above their corresponding nominal values.

3.2. Relationship of subproblem with First Order Second Moment methods

Structural reliability techniques have been applied successfully to solve certain class of stochastic programming problems [18], and also robust optimization design problems [6]. Trying to establish an analogy between subproblem (5) and the problem treated in [18], which attempts to optimize a given quantile of the random objective function, we could express the former as follows:

$$f^{\mathrm{S}}(\boldsymbol{d}) = \underset{\boldsymbol{y} \in \Omega(\boldsymbol{x}, \boldsymbol{d})}{\text{Minimum}} \quad (\boldsymbol{b}^T \ \boldsymbol{y}), \tag{30}$$

however, no information about the worst realization of nature is included yet, for that reason it is necessary to use the optimal value $\boldsymbol{d}^{\mathrm{ML}}$ of the uncertain parameters within the uncertainty set (27)-(29) which provide the

worst-minimum operational costs. It is obtained from solving the following problem:

$$q_\beta = \underset{\boldsymbol{d} \in \boldsymbol{D}(\beta)}{\text{maximum}} \quad [f^{\mathrm{S}}(\boldsymbol{d})] \ . \tag{31}$$

Let assume that $\boldsymbol{d}^{\mathrm{ML}}$ is known, in that case it is straightforward to calculate the optimal solution of subproblem (5) and values of the operating variables $\boldsymbol{y}^*$ as follows:

$$q_\beta = \underset{\boldsymbol{y} \in \Omega(\boldsymbol{x}, \boldsymbol{d})}{\text{Minimum}} \quad \boldsymbol{b}^T \ \boldsymbol{y}, \tag{32}$$

subject to

$$\boldsymbol{d} = \boldsymbol{d}^{\mathrm{ML}} : \boldsymbol{\eta} \tag{33}$$

where $\boldsymbol{\eta}$ is the dual variable vector associated with constraint (32), which provides information about how much the optimal objective function changes when uncertain parameters change around $\boldsymbol{d}^{\mathrm{ML}}$. For those values of the optimal operating variables $\boldsymbol{y}^*$, we could try to find a new value of the uncertain parameter vector in the vicinity of $\boldsymbol{d}^{\mathrm{ML}}$ but providing an even worst operating cost as follows:

$$\underset{\boldsymbol{d}}{\text{Maximize}} \quad q_\beta + \frac{\partial q_\beta}{\partial \boldsymbol{d}}^T (\boldsymbol{d} - \boldsymbol{d}^{\mathrm{ML}}) = q_\beta + \boldsymbol{\eta}^T (\boldsymbol{d} - \boldsymbol{d}^{\mathrm{ML}}) \tag{34}$$

subject to

$$\begin{aligned} \boldsymbol{d}^{\mathrm{G}} &\leq \bar{\boldsymbol{d}}^{\mathrm{G}} & (35) \\ \boldsymbol{d}^{\mathrm{D}} &\geq \bar{\boldsymbol{d}}^{\mathrm{D}}, & (36) \\ \boldsymbol{d} &= \bar{\boldsymbol{d}} + \boldsymbol{L}\boldsymbol{z}, & (37) \\ \left(\boldsymbol{z}^t \boldsymbol{z}\right)^{(1/2)} &\leq \beta. & (38) \end{aligned}$$

Since the ellipsoidal uncertainty set defines a convex region, the optimal solution $\boldsymbol{d}^{\mathrm{ML}}$ must be located at the boundary of the ellipsoidal uncertainty set, i.e. constraint (38) must hold as an equality constraint. The optimal solution of the random parameters from problem (34)-(38) corresponds to $\boldsymbol{d}^{\mathrm{ML}}$, otherwise it would mean that it is possible to obtain another value of the random parameters inside the ellipsoidal set providing a worst value of the operational cost, so that $\boldsymbol{d}^{\mathrm{ML}}$ would not correspond to the optimal solution, contradicting our initial hypothesis. This result might seem rather obvious,

however, this formulation (32)-(38) allows the proposal of a decomposition technique to look for the optimal solution of subproblem (5).

Before describing the algorithm, let transform the formulation (34)-(38) into the following equivalent problem at the optimum (see reference [18] for the formal proof):

$$\left.\begin{array}{rcl} \beta = \underset{\boldsymbol{d}}{\text{Minimum}} & & (\boldsymbol{z}^t\boldsymbol{z})^{(1/2)}, \\ \text{subject to} & & \\ \boldsymbol{d} & = & \bar{\boldsymbol{d}} + \boldsymbol{L}\boldsymbol{z}, \\ q_\beta & + & \dfrac{\partial q_\beta}{\partial \boldsymbol{d}}^T (\boldsymbol{d} - \boldsymbol{d}^{\mathrm{ML}}) \geq q_\beta. \end{array}\right\} \tag{39}$$

Note that the last constraint in (39) is the linear approximation of the equation:

$$g(\boldsymbol{x}, \boldsymbol{y}^*, \boldsymbol{d}) = f^{\mathrm{S}}(\boldsymbol{d}) \geq q_\beta \tag{40}$$

which avoids using operating variables and solving (30) to get (40). This problem structure is analogous to that given by [18], however in that work the objective function depends explicitly on the uncertain parameters and there is no need to use a first order approximation.

Problem (39) corresponds to the formulation associated with the structural reliability method FOSM [see 19, 20, 21, 22, 23, 24, 25] used to calculate the probability of obtaining an operational cost higher than q_β, i.e. $\mathrm{Prob}(f^{\mathrm{S}}(\boldsymbol{d}) > q_\beta)$. The random variables involved $\boldsymbol{d}$ belong to an n-dimensional space, which for given values of the decision $\boldsymbol{x}$ and operating $\boldsymbol{y}^*$ variables can be divided into two regions with respect to the limit-state equation $g(\boldsymbol{x}, \boldsymbol{y}^*, \boldsymbol{d}) = f^{\mathrm{S}}(\boldsymbol{d}) - q_\beta$, the *undesirable* ($\mathcal{U}$) and *desirable* ($\mathcal{D}$) regions. That is,

$$\begin{array}{l} \mathcal{U} \equiv \{\boldsymbol{d} | g(\boldsymbol{x}, \boldsymbol{y}^*, \boldsymbol{d}) = f^{\mathrm{S}}(\boldsymbol{d}) - q_\beta > 0\}, \\ \mathcal{D} \equiv \{\boldsymbol{d} || g(\boldsymbol{x}, \boldsymbol{y}^*, \boldsymbol{d}) = f^{\mathrm{S}}(\boldsymbol{d}) - q_\beta \leq 0\}. \end{array} \tag{41}$$

Note that once the q_β associated with the worst operational costs within the uncertainty set is known, obtaining a greater value of that cost associated with another realization of the uncertain parameters outside the uncertainty set is an undesirable situation.

In this context, the dependent normally distributed random variables $\boldsymbol{d}$ are transformed into independent standard normal random variables $\boldsymbol{z}$ that can be integrated straightforwardly using the linear Hasofer and Lind transformation [20], $\boldsymbol{L}$ is the Cholesky factorization of the variance-covariance

matrix, i.e. $\boldsymbol{\Sigma} = \boldsymbol{L}\boldsymbol{L}^T$. This factorization is always possible for positive definite matrices, and variance-covariance matrices must be positive definite by definition because their eigenvalues represent variances which must be always positive. This transformation is equivalent to the mapping transformation in (27). The optimal solution $\boldsymbol{z}^*$ corresponds to the closest point to the origin located on the limit-state equation in the standard and independent normal random space, β is the minimum distance so-called *reliability index* in the structural reliability scientific community, and $\boldsymbol{d}^* = \boldsymbol{d}^{\mathrm{ML}}$ is the *point of maximum likelihood*, i.e. the actual values of the uncertain parameters located on the limit-state equation where the probability is higher, and it represents the most likely values of the random parameters that produce a non desirable operational cost.

The final probability of obtaining an undesirable operational cost is related to the reliability index by the relation (see [6] or [18] for more details):

$$\mathrm{Prob}\left(f^{\mathrm{S}}(\boldsymbol{d}) > q_\beta\right) = \Phi(-\beta), \tag{42}$$

and the probability of a desirable operational cost is

$$\mathrm{Prob}\left(f^{\mathrm{S}}(\boldsymbol{d}) \leq q_\beta\right) = 1 - \Phi(-\beta) = \Phi(\beta), \tag{43}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal random variables. This method provides the exact probability if the limit-state equation is linear in the standard normal random space, i.e., if the resulting limit-state distribution is normally distributed, which is the case if uncertain parameters are normally distributed. Nevertheless, it is an approximation in case parameters do not follow a multivariate normal distribution, which is usually the case becasue we only use expectations $\bar{\boldsymbol{d}}$ and variance-covariance matrix $\boldsymbol{\Sigma}$. According to expressions (42) and (43), q_β corresponds to a upper quantile of the uncertain minimum operational cost random variable, or an approximation of the quantile, depending on whether the random parameters involved are normally distributed or not.

From this perspective, subproblem (5) can be interpreted as finding the quantile of the uncertain minimum operating costs given by expression (43). According to [18] the solution method proposed in this paper attempts to minimize the Value-at-Risk of total transmission costs.

Figure 1 shows a graphical illustration of the interpretation of subproblem 5 under FOSM method. Panel (a) shows the circumferences associated with

different transformed joint probability density function contours for the bi-dimensional case (two random variables involved), including that associated with the ellipsoidal uncertainty set (gray dashed line) whose distance from the origin is equal to β. It is also shown the operating cost contour equal to the quantile q_β, which is tangent with respect the ellipsoidal set circumference at the point of maximum likelihood. Note that combination of random variables whose operating costs are higher than q_β are in the undesirable region. If random variables are normally distributed, those operating cost contours are linear in the transformed space and the multidimensional problem can be reduced to one single dimension related to the standard normal random variable (see panel (b)). The probability of the operating costs to be inside the undesirable region, which requires a multidimensional integration can be calculated as $\Phi(-\beta)$ and it corresponds to the gray shadow area in panel (b). This probability can also be represented in the space of the operating cost random variable (see panel (b)), where $\Phi(-\beta)$ is equal to the shadow area associated with operating costs exceeding q_β.

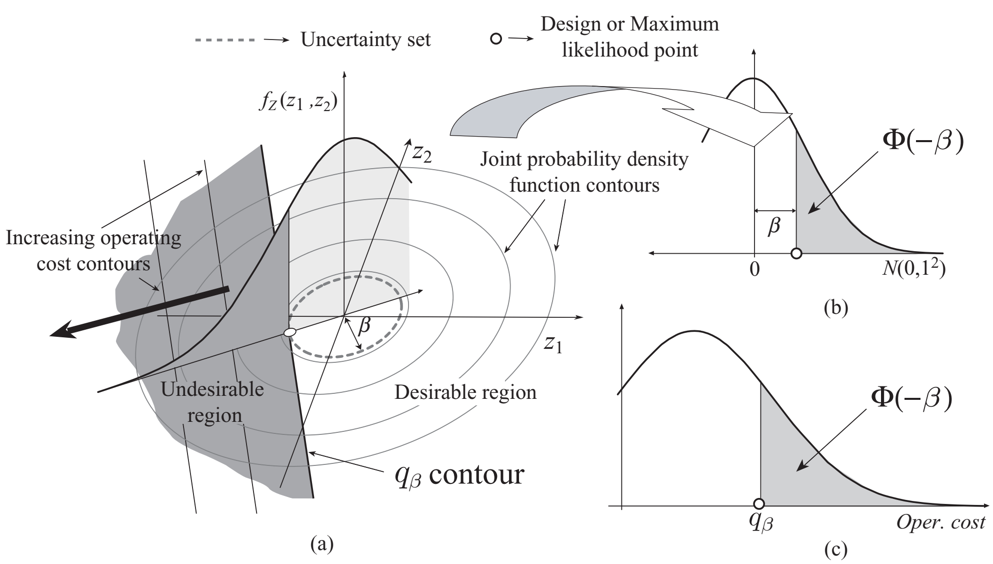

Figure 1: 3-D graphical interpretation of the FOSM method.

3.3. Proposed decomposition method

Once the uncertainty set is properly defined and the relationship between subproblem (5) and FOSM methods has been established, we focus on our methodological proposal for solving problem (1)-(3). It is based on iteratively solving the following problems:

Subproblem: Consisting on formulation (32)-(38), which is the equivalent with respect to (5) including the uncertainty set definition (27)-(29).

Master problem: It consist of the minimization of the objective function (17) subject to constraints (18)-(25).

This iterative scheme is known as "outer" because it tries to solve the complete TNEP problem. This outer iterative scheme have proved to be successful for solving TNEP problems using cardinality constrained uncertainty sets, however, the key for the success of the proposed methodology is in the resolution of the min-max subproblem. We propose a block coordinate descent algorithm [26] that takes advantage of sub-subproblems (32)-(33) and (34)-(38) to solve each of them optimizing the objective function over a subset of variables while the remaining variables are fixed to given values. This process is iteratively solved until no further improvement of the objective function is achieved. This iterative scheme is known as "inner". Convergence of this algorithm is only guaranteed for convex problems where each optimization sub-subproblem has a unique optimizer (strict convexity satisfies this requirement), which is the case for both sub-subproblems in this specific application.

Before proceeding to explain how the subproblem can be solved, the proposed outer iterative scheme is described step by step in the following algorithm, which is analogous to that proposed by [14] and in [15]:

Algorithm 3.1. (Outer iterative method).

Input: Selection of uncertainty factor β and the tolerance of the process ε. These data are selected by the decision maker.

Step 1: Initialization. Initialize the iteration counter to $\nu = 1$, and upper and lower bounds of the objective function $z^{(\mathrm{up})} = \infty$ and $z^{(\mathrm{lo})} = -\infty$.

Step 2: Solving the master problem at iteration ν. Solve the master problem (17) subject to constraints (18)-(25). The result provides values of the decision variables $\boldsymbol{x}_{(\nu)}$ and $\gamma_{(\nu)}$. Update the optimal objective function lower bound $z^{(\mathrm{lo})} = \boldsymbol{c}^T \boldsymbol{x}_{(\nu)} + \gamma_{(\nu)}$. Note that at the first iteration the optimal solution corresponds to the no investment case, alternatively, we could start with any other vector of decision variables.

Step 3: Solving subproblem at iteration ν. For given values of the decision variables $\boldsymbol{x}_{(\nu)}$, we calculate the worst operating costs within the uncertainty set or quantile $q_\beta^{(\nu)}$, obtaining also the corresponding uncertain parameters $\boldsymbol{d}_{(\nu)}^{\mathrm{ML}}$. This is achieved by solving sub-subproblems (32)-(33) and (34)-(38). Update the optimal objective function upper bound $z^{(\mathrm{up})} = \boldsymbol{c}^T \boldsymbol{x}_{(\nu)} + q_\beta^{(\nu)}$.

Step 4: Convergence checking. If $(z^{(\mathrm{up})} - z^{(\mathrm{lo})})/z^{(\mathrm{up})} \leq \varepsilon$ go to *Step 5*, else update the iteration counter $\nu \rightarrow \nu + 1$ and continue in *Step 2*.

Step 5: Output. The solution for a given tolerance corresponds to $\boldsymbol{x}^* = \boldsymbol{x}_{(\nu)}$.

■

Next step is the description of the method to solve the subproblem, as pointed out previously it consists on a block coordinate descent algorithm that takes advantage of formulations (32)-(33) and (34)-(38). The key is to chose an initial value of the uncertain parameter vector at iteration $\kappa = 1$, as candidate for maximum likelihood point $\boldsymbol{d}_{(\kappa)}^{\mathrm{ML}}$. We recommend the selection of low values of generation capacities and high values of demand loads because we know that the optimal maximum likelihood point is around this area of the uncertainty set. Once this selection is done, sub-subproblem (32)-(33) is solved replacing constraint (33) by:

$$\boldsymbol{d} = \boldsymbol{d}_{(\kappa)}^{\mathrm{ML}} : \boldsymbol{\eta}_{(\kappa)}. \tag{44}$$

The solution provides $q_\beta^{(\kappa)}$, the optimal values of operating variables $\boldsymbol{y}_{(\kappa)}^*$ for that particular realization of the random parameters and the derivatives of the corresponding worst operational cost with respect to the uncertain parameters, i.e. $\boldsymbol{\eta}_{(\kappa)}$. Next, we try update the point of maximum likelihood in a attempt to find either a point inside the ellipsoidal uncertainty set, this

case occurs if the initial point $\boldsymbol{d}^{\mathrm{ML}}_{(\kappa)}$ is outside the ellipsoidal set, or a point in the boundary of the ellipsoidal set with an even worst operational cost. This is done solving sub-subproblem (34)-(38) replacing the objective function (34) by:

$$\underset{\boldsymbol{d}}{\text{Maximize}} \quad q_{\beta}^{(\kappa)} + \boldsymbol{\eta}^T_{(\kappa)}(\boldsymbol{d} - \boldsymbol{d}^{\mathrm{ML}}_{(\kappa)}) \ . \tag{45}$$

The advantage of using this sub-subproblem is that [6] provides the analytical solution, which for this particular case is as follows:

$$\boldsymbol{d}^{\mathrm{ML}}_{(\kappa+1)} = \bar{\boldsymbol{d}} + \beta \frac{\boldsymbol{\Sigma}\boldsymbol{\eta}_{(\kappa)}}{\sqrt{\boldsymbol{\eta}^T_{(\kappa)}\boldsymbol{\Sigma}\boldsymbol{\eta}_{(\kappa)}}}. \tag{46}$$

This inner" iterative process continues until no further improvement in the worst operational cost is achieved, and/or the maximum likelihood point does not change between consecutive iterations. The corresponding algorithm is summarized as follows:

Algorithm 3.2. (Inner iterative method).

Input: The same input arguments than the outer algorithm are used and also the actual values of the decision variables from the outer problem $\boldsymbol{x}_{(\nu)}$, which are constants during this iterative process.

Step 1: Initialization. Initialize the iteration counter to $\kappa = 1$, and the initial values of the random parameters $\boldsymbol{d}^{\mathrm{ML}}_{(\kappa)}$.

Step 2: Determining the minimum operational cost at iteration κ. For given values of the random parameters $\boldsymbol{d}^{\mathrm{ML}}_{(\kappa)}$, obtain the operating variables providing the lowest operational costs for the actual iteration by solving problem (32)-(33) but replacing constraint (33) by (44). The result provides $q_{\beta}^{(\kappa)}$, the optimal values of operating variables $\boldsymbol{y}^*_{(\kappa)}$ for that particular realization of the random parameters and the derivatives of the corresponding worst operational cost with respect to the uncertain parameters, i.e. $\boldsymbol{\eta}_{(\kappa)}$.

Step 3: Convergence checking. If the iteration counter is equal to ($\kappa = 1$) go to *Step 4*, otherwise, proceed to check convergence, i.e., if $||\boldsymbol{d}^{\mathrm{ML}}_{(\kappa)} - \boldsymbol{d}^{\mathrm{ML}}_{(\kappa-1)}|| \leq \varepsilon$ go to *Step 5*, else continue in *Step 4*.

Step 4: Evaluating the new points of maximum likelihood. For given values of the operating variables $\boldsymbol{y}^*_{(\kappa)}$, we update the point of maximum likelihood using the analytical expression (46). Thus, updated representative values of the random parameters for the next iteration $\boldsymbol{d}^{\mathrm{ML}}_{(\kappa+1)}$ are obtained. The iteration counter must then be updated $\kappa \rightarrow \kappa + 1$ before proceeding to *Step 2*.

Step 5: Output. The solution for a given tolerance corresponds to $q^{(\kappa)}_{\beta}$, $\boldsymbol{y}^*_{(\kappa)}$ and $\boldsymbol{d}^{\mathrm{ML}}_{(\kappa)}$.

■

3.4. Bounded random variables

So far, the method developed in this paper deals with uncertainty coefficients by using the first and second probability moments associated with their probability distributions, and without assuming any particular type of distribution. This feature makes the method to be an approximation unless the random parameters truly follows a normal distribution. However, most of robust uncertainty sets proposed in the literature deal with random variables within a given interval. This information could be easily incorporated in the proposed method by including the following constraint set into the subsubproblem (34)-(38):

$$-\hat{\boldsymbol{d}} \leq \boldsymbol{d} - \bar{\boldsymbol{d}} \leq \hat{\boldsymbol{d}}. \tag{47}$$

These constraints constitute bounded convex polytopes both in the original space associated with random model parameters $\boldsymbol{d}$ and also in the transformed space $\boldsymbol{z}$. To solve this modified subproblem, the best strategy is as follows:

1. To use the analytical solution (46) and check afterwards whether constraints (47) hold, which means that the optimal solution has been achieved.
2. Otherwise, it means that the solution is not optimal according to those bounds, then the optimal solution is found by solving problem (34)-(38), including constraints (47) and removing the quadratic constraint (38). This results in a linear mathematical programming formulation.
3. Finally, in case the solution from item 2 is also out of the ellipsoidal uncertainty set, the complete problem (34)-(38) including constraints

(47) must be solved. This is a linear mathematical programming problem with one quadratic constraint that can be solve efficiently using interior point algorithms.

Alternatively, after checking the analytical solution from item 1, the conic problem in item 3 could be used directly instead of the linear problem in item 2.

4. Numerical case studies

In this section, we present numerical experiments of our model proposal and its comparison with the method proposed by [15], which proved to be the most computationally efficient method to solve the TNEP problem using cardinality constraint uncertainty sets. We use the same examples provided by [15], i.e. the Garver system illustrative example [27], and two realistic case studies: the IEEE 24-bus system [28] and the IEEE 118-bus test system [29]. To avoid providing redundant information numerical data will not be displayed in this manuscript, those readers interested in that information can find it in [15] and references therein.

All examples have been implemented and solved using GAMS [30, 31] and CPLEX 12, respectively, on a PC with four processors clocking at 2.39 GHz and 3.2 GB of RAM memory. Note that computing times reported correspond to the sum of running times used by the solvers in order to solve the masters and subproblems until the final solution is attained. We use the GAMS command `model.resusd`. We use the same tolerance for all problems and equal to $\epsilon = 10^{-6}$.

4.1. Illustrative example. Garver system

The proposed model is illustrated with the Garver 6-bus system, depicted in Figure 2. This system comprises 6 buses, 3 generators, 5 inelastic demands and 6 lines. Nominal values of generation capacities and demands and their offering and bidding prices can be found in [15]. The load-shedding cost is equal to the bid price of each demand. It is considered that a maximum of three lines can be installed between each pair of buses. Line data are obtained from Table I of [32], including construction costs, and the maximum available investment budget is €40 million.

The investment return period of each line is considered to be 25 years, and the discount rate is 10%, resulting in an annual amortization rate of 10%.

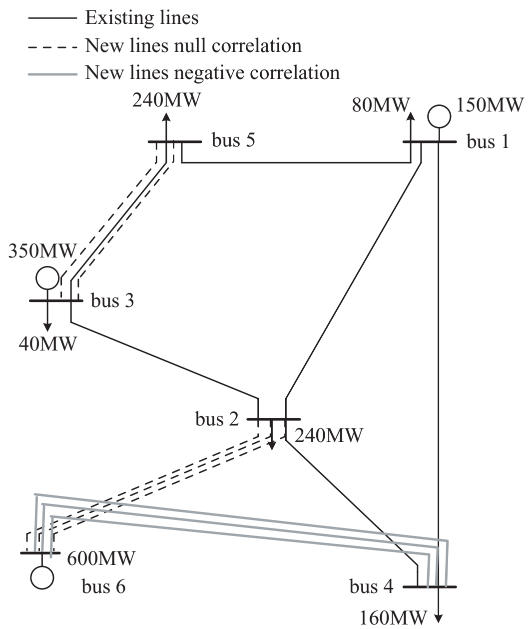

Figure 2: Garver's 6-bus test system.

Since investment cost is annualized, the weighted factor σ is equal to the number of hours of a year, i.e. 8760, obtaining an annualized load-shedding and power generation cost.

Regarding the uncertainty set, the standard deviations of power generation capacity and demand load are equal to 50% and 20% of their nominal values, respectively, divided by 2.3263. Note that we have selected this standard deviation so that in case the random parameters are normally distributed, the probability of those random parameters to be within the range defined by [15] is equal to 0.99. Initially the random parameters are assumed to be independent.

We have solved the robust TNEP problem for different reliability indexes β. These values have been selected in an attempt to find similar objective functions with respect to those obtained in [15] for different combinations of the uncertainty budgets Γ^{G} and Γ^{D}. In order to obtain statistically sound conclusions about computing times, we have solved the problem 100 times and obtained mean and standard deviation executions times. Results from this numerical experiments are given in Table 1.

Table 1: Computational results for Garver's 6-bus example

		[15] method				Proposed method				
	Optimal sol. (M€)	♯ iter.	Mean (s)	Std. (s)	β	Optimal sol. (M€)	♯ iter.	μ_t (s)	σ_t (s)
$\Gamma^{\mathrm{G}} = 3$ $\Gamma^{\mathrm{D}} = 5$	35505.31	3	0.644	0.054	6.58	35505.31	2	0.16	0.02
$\Gamma^{\mathrm{G}} = 2$ $\Gamma^{\mathrm{D}} = 3$	25832.22	4	0.936	0.094	4.3	25841.82	3	0.95	0.05
$\Gamma^{\mathrm{G}} = 1$ $\Gamma^{\mathrm{D}} = 2$	5861.92	4	1.011	0.072	2.6	5315.16	3	0.44	0.06
$\Gamma^{\mathrm{G}} = 0$ $\Gamma^{\mathrm{D}} = 0$	440.07	2	0.428	0.065	0	440.07	2	0.17	0.06

From this table the following observations are pertinent:

1. The proposed method is comparable in average computational time with respect to the method proposed by [15] for all cases, from the first and worst possible case corresponding to Soyster's solution [1] up to the deterministic case when uncertainty budgets are null. Regarding the standard deviation of computing times the values are also comparable.
2. Both methods require an small number of outer iterations.
3. The worst possible solution associated with $\beta = 6.58$ is the same than the analogous proposed by [15]. The reason is that all random parameters take their minimum and maximum possible values associated with z_i-values, i.e. equal to 2.3263. Note that $\sqrt{\sum_{i=1}^{8} 2.3263^2} = 6.58$.
4. The deterministic solution is also the same because it corresponds to the case where all uncertain parameters are equal to their nominal values.
5. For intermediate situations similar solutions have been found, which according to the reliability indexes $\beta = 4.3$ and $\beta = 2.6$ used and assuming normally distributed parameters, they correspond to quantiles $\Phi(\beta)$ of 0.99999146 and 0.99533881.

If we focus on the robust solution associated with the reliability index $\beta = 4.3$, the lines constructed are shown in Figure 2 in broken lines. At the point of maximum likelihood, only generation capacities of generators

at nodes 3 and 6, i.e those with higher capacities, reach their minimum values. The expansion investment is equal to 21.239M€. However, let assume that those power generators correspond to wind farm fields and their power productions are negatively correlated with coefficient -0.8, this makes the possibility of both generators at nodes 3 and 6 to be at their minimum capacities simultaneously remote. If we solve the robust problem under this new circumstance, the annualized optimal solution is reduced to 7739.721M€ with the same level of uncertainty, however, the cost of investment is higher 38.616M€ because besides the additional lines constructed in the previous case, three additional lines joining buses 4 and 6 are constructed (shown in gray continuous lines in Figure 2). In this case, at the point of maximum likelihood, only generation capacities of generators at nodes 1 and 3 reach their minimum values. This result proves that the correlation structure of generation capacities associated with renewable and intermittent sources is crucial for the appropriate expansion of the network.

Finally, we perform an additional simulation experiment using a reliability index $\beta = \Phi^{-1}(0.9) = 1.28155$ to check whether the optimal solution of the subproblem corresponds to the 0.9-quantile. Note that we have selected this level of protection because the maximum likelihood point at the optimal solution does not contain any limit of the random variables, in addition we divide the standard deviations of all random variables by 10 to make sure that we avoid sampling negative values of generations capacities and demand loads. This is important if we are interested in checking whether or not the optimal $q^*_\beta = 440.227$M€ for this example corresponds to the 0.9-quantile. We also consider the negative correlation between generation capacities at nodes 3 and 6. The procedure is simple, we sample random values of generation capacities and demand loads using the following expression:

$$\boldsymbol{d}^{\text{sim}} = \bar{\boldsymbol{d}} + \boldsymbol{L}\boldsymbol{z}^{\text{sim}} \tag{48}$$

where $\boldsymbol{z}^{\text{sim}}$ are randomly generated samples from an independent standard normal distribution. For each realization of the uncertain parameters, problem (32)-(33) replacing (33) by the expression $\boldsymbol{d} = \boldsymbol{d}^{\text{sim}}$ is solved, which allows calculating the minimum operating costs q^{sim}_β for those uncertain parameters. Repeating the process 1000 times, the probability of not exceeding the optimal value $q^*_\beta = 440.227$ is equal to the number of times the cost is lower or equal than q^*_β divided by the total number of samples. The probability obtained from Monte Carlo simulation is 0.907 which confirms that the optimal cost corresponds to the 0.9-quantile.

Figure 3 shows the histogram associated with operational cost simulation results, the normal distribution fitted and also the optimized quantile $q^*_\beta = 440.227$. Note that the cost distribution is normal, because uncertain parameters were supposed to be normally distributed. For this reason the quantile minimization is exact.

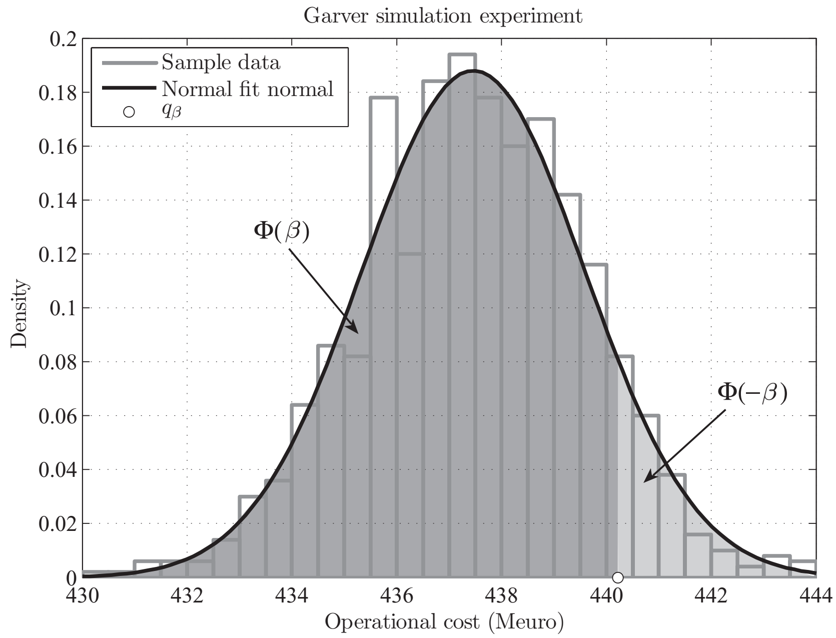

Figure 3: Simulation experiment for Garver's 6-bus test system.

Although this result is satisfactory, let us remind readers that this exact probabilities have been obtained under very specific circumstances which barely happen in practice, for the rest of cases the solution is an approximation of the quantile.

4.2. IEEE 24-bus Reliability Test System

The following case study described is based on the IEEE 24-bus Reliability Test System (RTS) [33], depicted in Fig. 4. The system comprises 24 buses, 34 existing corridors which can accept a maximum of three equal lines and 7 new corridors, 10 generating units and 17 loads. Data for lines in existing corridors are taken from [33], while line data for new corridors are obtained from Table I in [34]. Investment costs are €20 million, and analogously to

the Garver illustrative example, the investment return period of each line is considered to be 25 years, and the discount rate is 10%, resulting in an annual amortization rate of 10%. Data related to location of generators

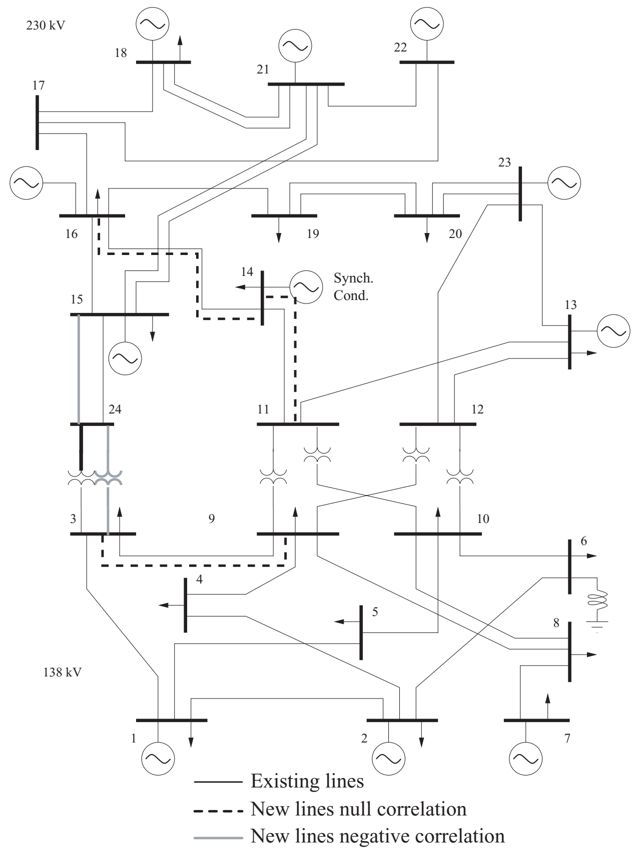

Figure 4: IEEE 24-bus reliability test system (RTS)

and demands in the network, maximum power offered, prices for generating units, power bid and the bid prices for demands can be found in [15]. The load-shedding cost is equal to 100 times the bid price of each demand.

Regarding the uncertainty set, power generation capacity can increase or decrease a maximum of 50% of their nominal values, while demand are allowed to change a maximum of 20%. Regarding the uncertainty set and analogously to the previous case, the standard deviations of power generation capacity and demand load are equal to 50% and 20% of their nominal values, respectively, divided by 2.3263. Note that we have selected this standard

deviation so that in case the random parameters are normally distributed, the probability of those random parameters to be within the range defined by [15] is equal to 0.99. Initially the random parameters are assumed to be independent.

We have solved the robust TNEP problem for different reliability indexes β. These values have been selected in an attempt to find similar objective functions with respect to those obtained in [15] for different combinations of the uncertainty budgets Γ^{G} and Γ^{D}. In order to obtain statistically sound conclusions about computing times, we have solved the problem 100 times and obtained mean and standard deviation executions times. Results from this numerical experiments are given in Table 2.

Table 2: Computational results for IEEE 24 RTS case study without correlation

			[15] method		Proposed method				
	Optimal sol. (M€)	♯ iter.	Mean (s)	Std. (s)	β	Optimal sol. (M€)	♯ iter.	μ_t (s)	σ_t (s)
$\Gamma^{\mathrm{G}} = 10$ $\Gamma^{\mathrm{D}} = 17$	500752.14	3	1.51	0.30	12.09	500752.14	2	1.34	0.19
$\Gamma^{\mathrm{G}} = 7$ $\Gamma^{\mathrm{D}} = 12$	454917.47	3	1.80	0.39	9	455488.75	3	2.35	0.71
$\Gamma^{\mathrm{G}} = 3$ $\Gamma^{\mathrm{D}} = 5$	348941.40	3	1.12	0.11	4.5	345282.84	3	0.78	0.10
$\Gamma^{\mathrm{G}} = 0$ $\Gamma^{\mathrm{D}} = 0$	219161.52	2	0.414	0.08	0	219161.52	2	0.28	0.06

From Table 2 we can extract the same conclusions than in the previous illustrative example:

1. The proposed method is comparable in average computational time with respect to the method proposed by [15] for all cases, from the first and worst possible case corresponding to Soyster's solution [1] up to the deterministic case when uncertainty budgets are null. Regarding the standard deviation of computing times the values are also comparable.
2. Both methods require an small number if outer iterations.
3. The worst possible solution associated with $\beta = 12.09$ is the same than the analogous proposed by [15]. The reason is that all random parame-

ters take their minimum and maximum possible values associated with z_i-values, i.e. equal to 2.3263. Note that $\sqrt{\sum_{i=1}^{27} 2.3263^2} = 12.09$.

4. The deterministic solution is also the same because it corresponds to the case where all uncertain parameters are equal to their nominal values.
5. For intermediate situations similar solutions have been found. For instance, the optimal solution associated with the reliability index $\beta = 4.5$, and assuming normally distributed parameters, corresponds to quantile $\Phi(\beta)$ of 0.999996602326875.

If we focus on the robust solution associated with the reliability index $\beta = 9$, the lines constructed are shown in Figure 4 in broken lines. Only three new lines are selected, from bus 3 to 9, from 11 to 14 and from 14 to 16. The investment cost is 14.307M€. At the point of maximum likelihood, there are several generation capacities reaching their minimum possible values, for instance generators 3 and 4 at buses 7 and 13, respectively. If we assume that those power generators correspond to wind farm fields and their power productions are negatively correlated with coefficient -0.8, this makes the possibility of both generators at nodes 3 and 4 to be at their minimum capacities simultaneously remote. If we solve the robust problem under this new circumstance, the annualized optimal solution is reduced to 432351.604M€ with the same level of uncertainty, however, the cost of investment is higher 16.606M€ and instead of constructing the lines from the previous case, only two lines from bus 3 to 24 and from 15 to 24 are selected as new lines (shown in gray continuous lines in Figure 4). This result also confirms that the correlation structure of generation capacities associated with renewable and intermittent sources is important for the appropriate expansion of the network.

Finally, we perform an additional simulation experiment using a reliability index $\beta = \Phi^{-1}(0.9) = 1.28155$ to check whether the optimal solution of the subproblem corresponds to the 0.9-quantile. Note that we have selected this level of protection because the maximum likelihood point at the optimal solution does not contain any limit of the random variables, which is the case with the standard deviation used in the example.

This is important if we are interested in checking whether or not the optimal $q^*_\beta = 251719.780$M€ for this example corresponds to the 0.9-quantile. We also consider the negative correlation between generation capacities at nodes 7 and 13. The procedure is analogous to that used in the previous

illustrative example. Repeating the process 1000 times, the probability of not exceeding the optimal value $q^*_\beta = 251719.780$ is equal to the number of times the cost is lower or equal than q^*_β divided by the total number of samples. The probability obtained from Monte Carlo simulation is 0.889 which confirms that the optimal cost corresponds to the 0.9-quantile.

Figure 5 shows the histogram associated with operational cost simulation results, the normal distribution fitted and also the optimized quantile $q^*_\beta = 251719.780$. Note that the cost distribution is normal, because uncertain parameters were supposed to be normally distributed. For this reason the quantile maximization is exact.

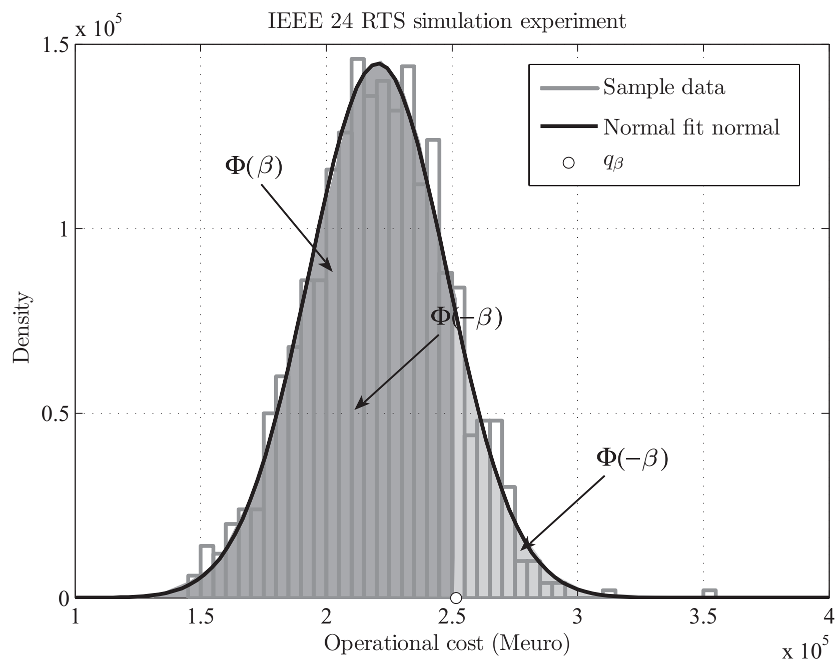

Figure 5: Simulation experiment for IEEE 24 RTS test system.

Although this result is satisfactory, let us remind readers that this exact probabilities have been obtained under very specific circumstances which barely happen in practice, for the rest of cases the solution is an approximation of the quantile.

4.3. IEEE 118-bus test system

Finally, we run additional computational tests using the IEEE 118-bus test system [29]. The system comprises 118 buses, 186 existing lines, 54 generating units and 91 loads. In addition, it is possible to construct up to 61 additional lines to duplicate each one of the following existing lines: 8, 12, 23, 32, 38, 41, 51, 68, 78, 96, 104, 118, 119, 121, 125, 129, 134, 159, 7, 9, 36, 117, 71, 131, 133, 147, 103, 65, 144, 168, 4, 13, 132, 69, 66, 67, 5, 89, 29, 167, 145, 70, 42, 90, 16, 174, 98, 99, 185, 93, 94, 128, 164, 97, 153, 146, 116, 163, 31, 92, 130. Data for lines in existing corridors are taken from [29]. Investment costs are €100 million, and analogously to both previous examples, the investment return period of each line is considered to be 25 years, and the discount rate is 10%, resulting in an annual amortization rate of 10%.

Data for generation capacities and demand loads are given in [15]. The load-shedding cost is equal to 1.2 times the bid price of each demand.

Regarding the uncertainty set, the standard deviations of power generation capacity and demand load are equal to 50% and 50% of their nominal values, respectively, divided by 2.3263. Note that we have selected this standard deviation so that in case the random parameters are normally distributed, the probability of those random parameters to be within the range defined by [15] is equal to 0.99. Random parameters are assumed to be independent.

We have solved the robust TNEP problem for different reliability indexes β. These values have been selected in an attempt to find similar objective functions with respect to those obtained in [15] for different combinations of the uncertainty budgets Γ^{G} and Γ^{D}. In order to obtain statistically sound conclusions about computing times, we have solved the problem 100 times and obtained mean and standard deviation executions times. Results from this numerical experiments are given in Table 3.

From Table 3 the same observations than in previous examples are obtained. However, in this case, the worst possible solution is associated with $\beta = 28.02$ because $\sqrt{\sum_{i=1}^{1} 452.3263^2} = 28.02$. Note that computational times are below 10 seconds, which prove the method to be applicable in real world energy transmission networks.

Table 3: Computational results for IEEE 118-bus test system

			[15] method		Proposed method				
	Optimal sol. (M€)	♯ iter.	Mean (s)	Std. (s)	β	Optimal sol. (M€)	♯ iter.	μ_t (s)	σ_t (s)
$\Gamma^{\mathrm{G}} = 54$ $\Gamma^{\mathrm{D}} = 91$	31994.36	4	7.73	0.80	28.02	31994.36	2	0.42	0.25
$\Gamma^{\mathrm{G}} = 35$ $\Gamma^{\mathrm{D}} = 60$	30032.93	5	53.46	60.36	21.0	29827.30	4	1.84	0.49
$\Gamma^{\mathrm{G}} = 15$ $\Gamma^{\mathrm{D}} = 20$	23352.97	4	24.26	26.05	9.5	21828.82	5	4.60	0.23
$\Gamma^{\mathrm{G}} = 0$ $\Gamma^{\mathrm{D}} = 0$	13929.23	2	0.56	0.07	0	13929.23	3	2.65	0.34

5. Conclusions

This paper extends the applicability of method [6], based on first-order second-moment methods from structural reliability and decomposition techniques, into the adaptive robust optimization problem associated with transmission network expansion planning. The method is specially suitable for problems where first and second order moments of the probability distributions of the uncertain parameters involved are available.

The proposed method has the following advantages:

1. The feasibility region defined by (27) is convex and only depends on variables $\boldsymbol{d}$.
2. The resulting sub-subproblem (32)-(33) for fixed values of the uncertain parameters is linear and it does not content binary variables, in contrast with respect to TNEP problems based on cardinality constrained uncertainty sets.
3. The sub-subproblem (34)-(38) resulting from fixing the operating variables to given values has analytical solution. In case additional bounds on uncertain parameters are included, the worst possible scenario requires solving a linear programming problem with one quadratic constraint, which can be solved efficiently using interior point algorithms.
4. The outer iterative method between subproblem and master problem provides upper and lower bounds of the objective function, which al-

lows checking if the problem is being solved appropriately during the iterative process. Besides, it converges in a small number of iterations.

5. This method allows including the correlation structure of the random variables involved easily, this is very important specially for wind power farms at different sites and demand loads, which might have significant correlations affecting the optimal expansion strategy.
6. The interpretation based on structural reliability allows selecting the level of conservationism in terms of approximate probabilities of exceeding the design operational costs, because this parameter allows selecting the quantile of the cost function to be optimized. This strategy is equivalent to the value-at-risk strategy widely used in financial and stochastic applications.

In summary, this paper proposes a new decomposition algorithm to attain the exact solution of the TNEP problem derived from using a two-stage adaptive robust strategy with ellipsoidal uncertainty sets. In addition, this procedure set the basis for alternative and more sophisticated robust uncertainty sets based on reliability methods. This would allow incorporating not only expected values and the correlation structure of uncertain parameters but appropriate marginal distributions of those parameters. This is a subject for further research.

Acknowledgements

This work has been partially funded by projects by Junta de Comunidades de Castilla-La Mancha, under project POII-2014-012-P; the Ministry of Science of Spain, under CICYT Project ENE2012-30679; and by the European Commission, under Grant Agreement Number 309048.

References

[1] A. L. Soyster, Convex programming with set-inclusive constraints and applications to inexact linear programming, Oper. Res. 21 (5) (1973) 1154–1157. doi:10.1287/opre.21.5.1154.

[2] A. Ben-Tal, A. Nemirovski, Robust convex optimization, Mathematics of Operations Research 23 (4) (1998) 769–805. doi:10.1287/moor.23.4.769.

[3] A. Ben-Tal, A. Nemirovski, Robust solutions of uncertain linear programs, Operations Research Letters 25 (1) (1999) 1 – 13. doi:http://dx.doi.org/10.1016/S0167-6377(99)00016-4.

[4] A. Ben-Tal, A. Nemirovski, Robust solutions of linear programming problems contaminated with uncertain data, Mathematical Programming 88 (3) (2000) 411–424. doi:10.1007/PL00011380.

[5] D. Bertsimas, M. Sim, The price of robustness, Oper. Res. 52 (1) (2004) 35–53. doi:10.1287/opre.1030.0065.

[6] R. Mínguez, Robust solutions of uncertain mixed-integer linear programs using structural-reliability and decomposition techniques, CoRR abs/1409.8593.
URL `http://arxiv.org/abs/1409.8593`

[7] G. B. Dantzig, Linear programming under uncertainty, Management Sci. 1 (1955) 197–206.

[8] A. Thiele, T. Terry, M. Epelman, Robust linear optimization with recourse, Ioe technical report tr09-01, University of Michigan (2010).

[9] D. Bertsimas, E. Litvinov, A. Sun, J. Zhao, T. Zheng, Adaptative robust optimization for the security constrained unit commitment problem, IEEE Trans. Power Syst. 28 (1) (2013) 52–63.

[10] P. Wu, H. Cheng, J. Xing, The interval minimum load cutting problem in the process of transmission network expansion planning considering uncertainty in demand, IEEE Trans. Power Syst. 23 (3) (2008) 1497–1506.

[11] H. Yu, Y. Chung, K. P. Wong, Robust transmission network expansion planning method with taguchi's orthogonal array testing, IEEE Trans. Power Syst. 26 (3) (2011) 1573–1580.

[12] R. A. Jabr, Robust transmission network expansion planning with uncertain renewable generation and loads, IEEE Trans. Power Syst. 28 (4) (2013) 4558–4567.

[13] B. Cheng, J. Wang, L. Wang, Y. He, Z. Wang, Robust optimization of transmission expansion planning: Minimax cost vs. minimax regret, IEEE Trans. Power Syst. (2014) 1–9doi:10.1109/TPWRS.2014.2313841.

[14] C. Ruiz, A. J. Conejo, Robust transmission expansion planning, European Journal of Operational Research (2014) 1–12doi:http://dx.doi.org/j.ejor.2014.10.030.

[15] R. Mínguez, R. García-Bertrand, Robust transmission network expansion planning in energy systems: Improving computational performance, CoRR abs/1501.05480.
URL `http://arxiv.org/abs/1501.05480`

[16] B. Zeng, L. Zhao, Solving two-stage robust optimization problems using a column-and-constraint generation method, Operations Research Letters 41 (5) (2013) 457–461.

[17] R. A. Jabr, Robust self-scheduling under price uncertainty using conditional value-at-risk, IEEE Trans. Power Syst. 20 (4) (2005) 1852–1858.

[18] R. Mínguez, A. J. Conejo, R. García-Bertrand, Reliability and decomposition techniques to solve certain class of stochastic programming problems, Reliability Engineering & System Safety (96) (2011) 314–323. doi:10.1016/j.rees.2010.09.011.

[19] A. M. Freudenthal, Safety and the probability of structural failure, Transactions, ASCE 121 (1956) 1337–1397.

[20] A. M. Hasofer, N. C. Lind, Exact and invariant second moment code format, J. Engrg. Mech. 100 (EM1) (1974) 111–121.

[21] R. Rackwitz, B. Fiessler, Structural reliability under combined load sequences, Comput. Struct. 9 (1978) 489–494.

[22] O. Ditlevsen, Principle of normal tail approximation, J. Engineering Mechanics Div., ASCE 107 (6) (1981) 1191–1208.

[23] M. Hohenbichler, R. Rackwitz, Non-normal dependent vectors in structural safety, J. Engineering Mechanics Div., ASCE 107 (6) (1981) 1227–1238.

[24] R. E. Melchers, Structural reliability analysis and prediction, 2nd Edition, John Wiley & Sons, New York, 1999.

[25] R. Mínguez, M. Castillo, Reliability-based optimization in engineering using decomposition techniques and FORMS, Structural Safety 31 (3) (2009) 214–223.

[26] A. J. Conejo, E. Castillo, R. Mínguez, R. García-Bertrand, Decomposition techniques in mathematical programming. Engineering and science applications, Springer-Verlag Berlin Heidelberg, New York, 2006.

[27] L. L. Garver, Transmission network estimation using linear programming, IEEE Trans. Power Apparat. Syst. 89 (7) (1970) 1688–1697.

[28] R. T. S. T. Force, The IEEE Reliability Test System-1996, IEEE Trans. Power Syst. 14 (3) (1979) 1010–1020.

[29] IEEE 118-bus system, Available: http://motor.ece.iit.edu/ data/data 118 bus.pdf.

[30] http://www.gams.com.

[31] A. Brooke, D. Kendrick, A. Meeraus, R. Raman, GAMS: A user's guide, GAMS Development Corporation, Washington, 1998.

[32] L. P. Garcés, A. J. Conejo, R. García-Bertrand, R. Romero, A bi-level approach to transmission expansion planning within a market environment, IEEE Trans. Power Syst. 24 (3) (2009) 1513–1522.

[33] Reliability Test System Task Force, The IEEE Reliability Test System 1996, IEEE Transactions on Power Systems 14 (3) (1999) 1010–1020.

[34] R. Fang, D. J. Hill, A new strategy for transmission expansion in competitive electricity markets, IEEE Trans. Power Syst. 18 (1) (2003) 374–380.